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High-Symmetry point groups in nuclear structure and their experimental manifestations
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- Phys. Rev. C 105, 034348 (2022).
- Phys. Rev. C 106, 054314 (2022).
- Phys. Rev. C 107, 054304 (2023)

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Symmetry

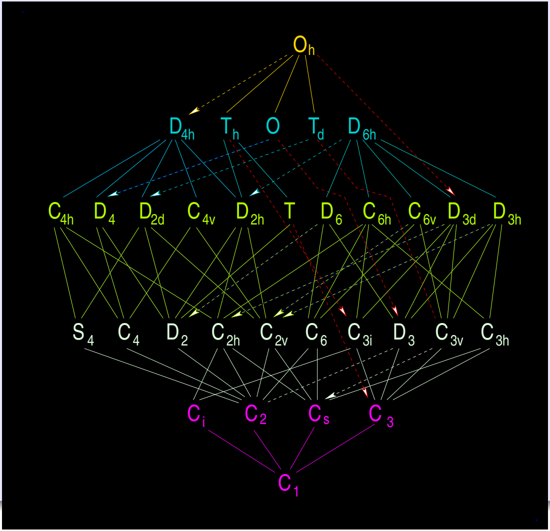
Symmetry is relative to our knowledge and technical possibilities to distinguish physical objects.

Let "$\sim$" be an equivalence relation distinguishing physical objects belonging to the set X, then the symmetry S of an object $\mathcal{O}$ is the one-to-one transformation $\hat{S} : X \to X$:

$$\hat{S}\mathcal{O} \sim \mathcal{O} \ ,$$

i.e. $\mathcal{O}$ is invariant in respect to the transformation $\hat{S}$.

32 point groups (without icosahedral group)



Symmetry of a Hamiltonian $\hat{H}$

$\mathrm{Sym}(\hat{H}) = \mathrm{G}$:

$$\text{For all } g \in \mathrm{G} \;\Rightarrow\; \hat{g}\hat{H}\hat{g}^{-1} = \hat{H}.$$

- Ireducible representations: degeneracy of the energy spectrum.
- Equivalent representations. Decomposition of the state space into invariant subspaces (multiplicity quantum numbers).
- Selection rules.
- Wigner-Eckart theorem.

Degeneracy of energy spectrum

Energy spectrum degeneracy of $\hat{H}$ with a symmetry G:

$$\hat{H}\,|\nu\Gamma a\rangle = E_{\nu\Gamma}\,|\nu\Gamma a\rangle$$
$$\hat{H}\,\hat{g}|\nu\Gamma a\rangle = E_{\nu\Gamma}\,\hat{g}|\nu\Gamma a\rangle$$

for all $g \in \mathrm{G}$, $\nu = 1, 2, \ldots, n_\Gamma$ (multiplicity), $a = 1, 2, \ldots, \dim[\Gamma]$.

For fixed ν and Γ the subspace $\mathrm{Lin}\{\hat{g}|\nu\Gamma a\rangle : g \in \mathrm{G}\}$ is an invariant irreducible subspace $\mathcal{K}_{\nu\Gamma}$ of $\mathcal{K}$, i.e.,

$$\mathcal{K} = \bigoplus_{\Gamma} \bigoplus_{\nu=1}^{n_\Gamma} \mathcal{K}_{\nu\Gamma}$$

Degeneracy s_Γ (it means $a = 1, 2, \ldots, s_\Gamma$) of the energy spectrum $\{E_{\nu\Gamma}\}$ is equal to the dimension of i.r. $[\Gamma]$, i.e., $s_\Gamma = \dim(\mathcal{K}_{\nu\Gamma})$.

Structure of any Hamiltonian $\hat{H}$

Spectral decomposition of a general $\hat{H}$

$$\hat{H} = \sum_{\rho} E_{\rho} |\rho\rangle\langle\rho|$$

ρ represents a set of required quantum numbers.

3D Harmonic oscillator

$$\hat{H} = \sum_{N} \hbar\omega(N + \frac{3}{2}) \sum_{L,M} |NLM\rangle\langle NLM|$$

Spectrum degeneracy: $s_N =$(number of allowed pairs (L, M) for fixed N),
In the chain SU(3) $\subset$ SO(3) no multiplicity n_L higher than 1, $L = N, N-2, \ldots$

3D Harmonic oscillator – modified

3D Harmonic oscillator with rotational spectrum

$$\hat{H} = \sum_L \hbar^2 L(L+1) \sum_N \left(\sum_M |NLM\rangle\langle NLM| \right)$$

Spectrum degeneracy:
s_L =(number of allowed pairs (N, M) for fixed L),
Multiplicity $n_L = \infty$

Bohr Hamiltonian $\hat{H}$

Bohr Hamiltonian 5D Harmonic oscillator

$$\hat{H} = \sum_N \hbar\omega(N + \frac{5}{2}) \sum_{v,n_\Delta,L,M} |Nvn_\Delta LM\rangle\langle Nvn_\Delta LM|$$

Bohr Hamiltonian: N, L-dependent spectrum

$$\hat{H} = \sum_{NL} E_{NL} \sum_{v,n_\Delta} \left(\sum_M |Nvn_\Delta LM\rangle\langle Nvn_\Delta LM| \right)$$

Spectrum degeneracy:
$s_{(N,L)}$ =(number of allowed triplets (v, n_Δ, M) for fixed N, L).
Multiplicity:
n_L=(number of allowed pairs (v, n_Δ) for fixed N, L) – n_L states with the angular momentum L is observed.

Degeneracy of the energy spectrum

Time reversal can change degeneracy

Rotations are not invariant with respect to the time reversal operation $\Rightarrow$ the points groups are affected by the time reversal.

Wigner: Three types of representations:

I $[\Gamma] = [\Gamma^*]$ (real).

II $[\Gamma]$ is complex and **not** equivalent to $[\Gamma^*]$.

III $[\Gamma]$ is complex and equivalent to $[\Gamma^*]$ but cannot be made real.

For even-even nuclei there are only types I and II

I No additional degeneracy due to time reversal.

II The degeneracy is doubled (Kramer's theorem).

C_4

(9) C_4. This is a cyclic group consisting of the identity; a rotation through 90°, C_4; through −90°, C_4^{-1}; and through 180°, C_2 all about a given axis (taken as z).

(10) S_4. This is a cyclic group, isomorphic to C_4, consisting of the identity; the rotation reflection $S_4 = IC_4^{-1}$; a twofold rotation, C_2; and the operator $S_4^{-1} = IC_4$. All rotations are taken about the z axis.

Table 25. Character Table and Basis Functions for the Groups C_4 and S_4

C_4	E	$\bar{E}$	C_4	$\bar{C}_4$	C_2	$\bar{C}_2$	C_4^{-1}	$\bar{C}_4^{-1}$	Time Inv.	Bases for C_4	Bases for S_4
S_4	E	$\bar{E}$	S_4^{-1}	$\bar{S}_4^{-1}$	C_2	$\bar{C}_2$	S_4	$\bar{S}_4$			
Γ_1	1	1	1	1	1	1	1	1	a	z or S_z	S_z
Γ_2	1	1	−1	−1	1	1	−1	−1	a	xy	z or xy
Γ_3	1	1	i	i	−1	−1	−i	−i	b	$-i(x+iy)$ or $-(S_x+iS_y)$	$-(S_x+iS_y)$ or $i(x-iy)$
Γ_4	1	1	−i	−i	−1	−1	i	i	b	$i(x-iy)$ or (S_x-iS_y)	(S_x-iS_y) or $-i(x+iy)$
Γ_5	1	−1	ω	$-\omega$	i	−i	$-\omega^3$	ω	b	$\phi(1/2, 1/2)$	$\phi(1/2, 1/2)$

C_4

Table 29. Full Rotation Group Compatibility Table for the Group C_4

$D_0^{\pm}$	Γ_1
$D_1^{\pm}$	$\Gamma_1 + \Gamma_3 + \Gamma_4$
$D_2^{\pm}$	$\Gamma_1 + 2\Gamma_2 + \Gamma_3 + \Gamma_4$
$D_3^{\pm}$	$\Gamma_1 + 2\Gamma_2 + 2\Gamma_3 + 2\Gamma_4$
$D_4^{\pm}$	$3\Gamma_1 + 2\Gamma_2 + 2\Gamma_3 + 2\Gamma_4$
$D_5^{\pm}$	$3\Gamma_1 + 2\Gamma_2 + 3\Gamma_3 + 3\Gamma_4$
$D_6^{\pm}$	$3\Gamma_1 + 4\Gamma_2 + 3\Gamma_3 + 3\Gamma_4$
$D_{1/2}^{\pm}$	$\Gamma_5 + \Gamma_6$
$D_{3/2}^{\pm}$	$\Gamma_5 + \Gamma_6 + \Gamma_7 + \Gamma_8$
$D_{5/2}^{\pm}$	$\Gamma_5 + \Gamma_6 + 2\Gamma_7 + 2\Gamma_8$
$D_{7/2}^{\pm}$	$2\Gamma_5 + 2\Gamma_6 + 2\Gamma_7 + 2\Gamma_8$
$D_{9/2}^{\pm}$	$3\Gamma_5 + 3\Gamma_6 + 2\Gamma_7 + 2\Gamma_8$
$D_{11/2}^{\pm}$	$3\Gamma_5 + 3\Gamma_6 + 3\Gamma_7 + 3\Gamma_8$
$D_{13/2}^{\pm}$	$3\Gamma_5 + 3\Gamma_6 + 4\Gamma_7 + 4\Gamma_8$

T_d and O

Table 85. Full Rotation Group Compatibility Table for the Group O

$D_0^\pm$	Γ_1
$D_1^\pm$	Γ_4
$D_2^\pm$	$\Gamma_3 + \Gamma_5$
$D_3^\pm$	$\Gamma_2 + \Gamma_4 + \Gamma_5$
$D_4^\pm$	$\Gamma_1 + \Gamma_3 + \Gamma_4 + \Gamma_5$
$D_5^\pm$	$\Gamma_3 + 2\Gamma_4 + \Gamma_5$
$D_6^\pm$	$\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + 2\Gamma_5$
$D_{1/2}^\pm$	Γ_6
$D_{3/2}^\pm$	Γ_8
$D_{5/2}^\pm$	$\Gamma_7 + \Gamma_8$
$D_{7/2}^\pm$	$\Gamma_6 + \Gamma_7 + \Gamma_8$
$D_{9/2}^\pm$	$\Gamma_6 + 2\Gamma_8$
$D_{11/2}^\pm$	$\Gamma_6 + \Gamma_7 + 2\Gamma_8$
$D_{13/2}^\pm$	$\Gamma_6 + 2\Gamma_7 + 2\Gamma_8$

Table 86. Full Rotation Group Compatibility Table for the Group T_d

D_0^+	Γ_1	D_0^-	Γ_2
D_1^+	Γ_4	D_1^-	Γ_5
D_2^+	$\Gamma_3 + \Gamma_5$	D_2^-	$\Gamma_3 + \Gamma_4$
D_3^+	$\Gamma_2 + \Gamma_4 + \Gamma_5$	D_3^-	$\Gamma_1 + \Gamma_4 + \Gamma_5$
D_4^+	$\Gamma_1 + \Gamma_3 + \Gamma_4 + \Gamma_5$	D_4^-	$\Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5$
D_5^+	$\Gamma_3 + 2\Gamma_4 + \Gamma_5$	D_5^-	$\Gamma_3 + \Gamma_4 + 2\Gamma_5$
D_6^+	$\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + 2\Gamma_5$	D_6^-	$\Gamma_1 + \Gamma_2 + \Gamma_3 + 2\Gamma_4 + \Gamma_5$

"Accidental" degeneracy of energy spectrum

Assume, the quantum numbers ν can be split into two sets
$\nu = (\nu', \nu'')$,
where $\nu' = (\nu_1, \nu_2, \ldots, \nu_s)$ and $\nu'' = (\nu_{s+1}, \nu_{s+2}, \ldots, \nu_r)$.
Energy spectrum degeneracy of $\hat{H}$ with a symmetry G:

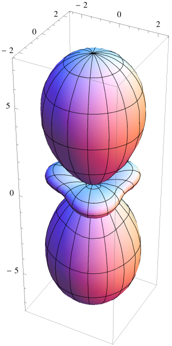
$$\hat{H} = \sum_{\Gamma} \sum_{\nu''} E_{\nu'',\Gamma} \sum_{\nu'} \left(\sum_{a} |\nu', \nu'', \Gamma, a\rangle\langle\nu', \nu'', \Gamma, a| \right)$$

"Multiplicity" degeneracy

Observed accidental degeneracy of the energy level $E_{\nu'',\Gamma}$
$=$ number of elements $\nu' = (\nu_1, \nu_2, \ldots, \nu_s)$ for fixed Γ and
$\nu'' = (\nu_{s+1}, \nu_{s+2}, \ldots, \nu_r)$.

Partial Symmetries

Two symmetries in one body – partial symmetries 1/2



Nuclear surface: $\alpha_{20} = 10 \rightarrow \overline{\mathrm{SO}(2)}$; $\alpha_{33} = 0.5 \rightarrow \overline{\mathrm{C}}_3$

$$R(\{\alpha\};\theta,\phi) = R_0(1 +$$
$$+\alpha^{\star}_{20}Y_{20}(\theta,\phi) + \alpha^{\star}_{33}Y_{22}(\theta,\phi) + \alpha^{\star}_{3,-3}Y_{3,-3}(\theta,\phi))$$

Partial–symmetries, non-orthogonal decomposition 2/2

The schematic quadrupole+octupole model Hamiltonian:

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{vib} + \hat{\mathcal{H}}_{rot}$$

$$\hat{\mathcal{H}}_{vib} = \hat{\mathcal{H}}_{vib;2}(\alpha_2) + \hat{\mathcal{H}}_{vib;3}(\alpha_3)$$
$$\hat{\mathcal{H}}_{rot} = \hat{\mathcal{H}}_{rot}(\Omega)$$

If the Hamiltonian is related to the above nuclear shape:

$$\mathrm{Sym}(\hat{\mathcal{H}}_{vib;2}) = \overline{\mathrm{SO}(2)}_{vib} \quad \mathrm{Sym}(\hat{\mathcal{H}}_{vib;3}) = \overline{\mathrm{C}}_{3;vib} \quad \mathrm{Sym}(\hat{\mathcal{H}}_{rot}) = \overline{\mathrm{G}}_{rot}$$

Open problem: partial selection rules.

Partial–symmetries, orthogonal decomposition

Spectral theorem

Assume the discrete spectrum of $\hat{\mathcal{H}}$, then:

$$\hat{\mathcal{H}} = \sum_{\nu} \epsilon_\nu P_\nu$$

Notation:
A) The operator A has the symmetry G:

$$\mathrm{G} = \mathrm{Sym}(A)$$

B) Collection of the projectors P_ν having the same symmetry G:

$$\mathcal{O}_{\mathrm{G}} = \{P_\nu : \mathrm{Sym}(P_\nu) = \mathrm{G}\}$$

Partial–symmetries, orthogonal decomposition

The partial Hamiltonians:

$$\hat{\mathcal{H}}_{\mathrm{G}} = \sum_{P_\nu \in \mathcal{O}_{\mathrm{G}}} \epsilon_\nu P_\nu .$$

$\hat{\mathcal{H}}_{\mathrm{G}}$ has the symmetry G.

Orthogonal decomposition of $\hat{\mathcal{H}}$ into the partial Hamiltonians:

$$\hat{\mathcal{H}} = \sum_{\mathrm{G}} \hat{\mathcal{H}}_{\mathrm{G}}$$

$\mathrm{G} \neq \mathrm{G}' \Rightarrow$

$$\hat{\mathcal{H}}_{\mathrm{G}} \hat{\mathcal{H}}_{\mathrm{G}'} = 0 \qquad (*)$$

Eigenproblem

To solve the eigenequation for $\hat{\mathcal{H}} = \sum_{\mathrm{G}} \hat{\mathcal{H}}_{\mathrm{G}}$ it is sufficient to solve the eigenproblems for all partial Hamiltonians:

$$\hat{\mathcal{H}}_{\mathrm{G}} |\mathrm{G}; \mu\Gamma a\rangle = \epsilon^{\mathrm{G}}_{\mu\Gamma} |\mathrm{G}; \mu\Gamma a\rangle.$$

By definition, for $\mathrm{G}' \neq \mathrm{G}$

$$\hat{\mathcal{H}}_{\mathrm{G}'} |\mathrm{G}; \mu\Gamma a\rangle = 0.$$

Here: μ labels the equivalent i.r. of the group G. We get

$$\hat{\mathcal{H}} |\mathrm{G}; \mu\Gamma a\rangle = \epsilon^{\mathrm{G}}_{\mu\Gamma} |\mathrm{G}; \mu\Gamma a\rangle.$$

and reversely.

Example: The vibrator+rotor Hamiltonian

The Hamiltonian (all is in the intrinsic frame):

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{vib} + \sum_{l=1}^{3} A(\hat{n}_l)\hat{J}_l^2,$$

$$\hat{\mathcal{H}}_{vib} = \hbar\omega \sum_{l} \hat{n}_l,$$

where $\hat{n}_l$= number of phonon operators in $l = 1, 2, 3$ directions, $\hat{J}_l$ are angular momentum operators.
The vibrations and rotations are independent:

$$[\hat{n}_l, \hat{J}] = 0, \text{ for all } l = 1, 2, 3.$$

Definition of the logical function δ:

$$\delta(Q) = \begin{cases} 1 & \text{if } Q\text{=True}, \\ 0 & \text{if } Q\text{=False}. \end{cases}$$

Sub-Hamiltonians 1/2

The sub-Hamiltonians of $\hat{\mathcal{H}}$ (laboratory symmetries omitted):

$$\hat{\mathcal{H}}_{\mathrm{O}(3)} = \delta(\hat{n}_1 = \hat{n}_2 = \hat{n}_3)\left(\hat{\mathcal{H}}_{vib} + A(\hat{n}_3)\hat{J}^2\right)$$

$$\hat{\mathcal{H}}_{\mathrm{O}(2)_{l_1}} = \delta(\hat{n}_{l_2} = \hat{n}_{l_3})\delta(\hat{n}_{l_1} \neq \hat{n}_{l_2})\Big(\hat{\mathcal{H}}_{vib} + $$
$$+ A(\hat{n}_{l_1})\hat{J}_{l_1}^2 + A(\hat{n}_{l_2})(\hat{J}_{l_2}^2 + \hat{J}_{l_3}^2)\Big)$$

$$\hat{\mathcal{H}}_{\mathrm{D}_{2h}} = \delta(\hat{n}_1 \neq \hat{n}_2 \neq \hat{n}_3 \neq \hat{n}_1)\left(\hat{\mathcal{H}}_{vib} + \sum_{l=1}^{3} A(\hat{n}_l)\hat{J}_l^2\right)$$

For $\hat{\mathcal{H}}_{\mathrm{O}(2)_{l_1}}$, $l_1 \neq l_2 \neq l_3 \neq l_1$, where $l_1, l_2, l_3 = 1, 2, 3$.

The symmetry of $\hat{\mathcal{H}}_{vib}$ is fixed $= \overline{\mathrm{SU}(3)}$.

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\mathrm{O}(3)} + \sum_{l=1}^{3} \hat{\mathcal{H}}_{\mathrm{O}(2)_l} + \hat{\mathcal{H}}_{\mathrm{D}_{2h}},$$

Sub-Hamiltonians 2/2

- The eigenproblem of the sub-Hamiltonians:

$$\hat{\mathcal{H}}_{\mathrm{G}}|[\mathrm{G}]n_1n_2n_3;JM\mu\rangle = \epsilon^{\mathrm{G}}_{n_1n_2n_3;J\mu}|[\mathrm{G}]n_1n_2n_3;JM\mu\rangle.$$

  The eigenvaules and eigenvectors solve the eigenproblem of the full Hamiltonian $\hat{\mathcal{H}}$.

- The sub-Hamiltonians for the symmetries O(3) and O(2) have analytical solutions:

$$\phi_\nu(\alpha,\Omega) \equiv \phi_{n_1n_2n_3;JMK}(\alpha,\Omega) = \left(\Pi_{l=1}^{3} u_{n_l}(\alpha_l)\right) r^{J}_{MK}(\Omega),$$

  where $u_n(b,\alpha)$ are 1-D harmonic oscillator functions, $b = \sqrt{m\omega/\hbar}$ is the h.o. length, $r^J_{MK}(\Omega)$ are complex conjugated and normalized Wigner functions for SO(3), $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

EMG transitions

Clebsch-Gordan series and coefficients (multiplicities):

$$\Delta^{\Gamma_1} \times \Delta^{\Gamma_2} \sim \bigotimes_{\Gamma} n^{\Gamma}_{\Gamma_1\Gamma_2} \Delta^{\Gamma}$$

$$\Psi^{\Gamma,\alpha}_{c} = \sum_{a=1}^{\dim(\Gamma_1)} \sum_{b=1}^{\dim(\Gamma_2)} (\Gamma_1 a \Gamma_2 b | \Gamma c; \alpha) \phi^{\Gamma_1}_{a} \xi^{\Gamma_2}_{b}$$

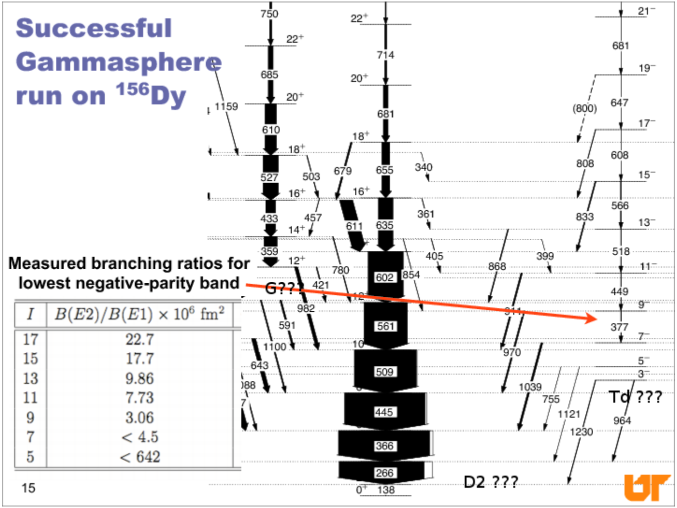
Irreducible tensor for a group G

$$\hat{g} Q^{\Gamma}_{a} \hat{g}^{-1} = \sum_{k=1}^{\dim(\Gamma)} \Delta^{\Gamma}_{ka}(g) Q^{\Gamma}_{k}$$

Wigner-Eckart theorem:

$$\langle \phi^{\Gamma}_{a} | Q^{\Gamma_1}_{k} | \xi^{\Gamma_2}_{b} \rangle = \sum_{\alpha}^{n^{\Gamma}_{\Gamma_1\Gamma_2}} (\Gamma a \Gamma_1 b | \Gamma_2 l; \alpha)^{\star} \langle \phi^{\Gamma} || Q^{\Gamma_1} || \xi^{\Gamma_2} \rangle_{\alpha}$$

Experiment Argone 2009, spectrum ^{156}Dy



I	$B(E2)/B(E1) \times 10^6$ fm^2
17	22.7
15	17.7
13	9.86
11	7.73
9	3.06
7	< 4.5
5	< 642

Figure: Spectrum ^{156}Dy (Lee Riedinger)

Pure octupole model – collective Eλ transitions

IF *the Euler angles are chosen to fix octupoles in the principal axes frame.*
For pure octupole T$_d$ collective model ($\bar{\alpha}_{3\mu} = 0$ for $\mu \neq \pm 2$) the operators:

- $Q^{lab}_{1\mu} = 0$, because of $(3030|10) = 0$.
- $Q^{lab}_{2\mu} = 0$, because of $(323-2|20) = 0$.

The only non-zero moment is the octupole one.

Problems

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